Journal of Engineering Physics and Thermophysics, Vol. 68, No. 4, 1995

LINEAR THERMODYNAMIC SYSTEMS WITH MEMORY. 1. BASIC TENETS OF THE THERMODYNAMIC THEORY

V. L. Kolpashchikov and A. I. Shnip

UDC 536.7

The article starts a series of investigations of generalized linear thermodynamic systems with memory. We formulate the basic tenets of the nonequilibrium thermodynamic theory of such systems. The second law is given in the form of a postulate generalizing the classical formulation that requires that the integral of reduced heat have a fixed sign in any cyclic process. We derived some auxiliary results necessary for the main theorem to be proved.

Introduction. Contemporary thermodynamics is distinguished by a wide variety of schools, approaches, and methods [1-12]. They differ not only in the objects of investigation, initial premises, levels of mathematical rigor, but also in the objectives placed before the thermodynamic theory. According to the tradition going back to the classical thermodynamics of irreversible processes of the Brussels school [1, 2], the aim of the thermodynamic theory of continuum is the construction of the field equations of a continuous medium, and this aim is retained in many modern trends of thermodynamics [4, 7, 8]. Actually, such a construction consists only in the derivation of material, or, in the present-day language, constitutive equations, which, together with the equations for the laws of conservation in a local form, form a closed system of field equations. However, any specific system of equations describes only a particular medium (more precisely, a narrow class of media) and therefore cannot be obtained only by the methods of such a general theory as thermodynamics. This means that to accomplish a specified task one is bound to use assumptions lying outside the scope of pure thermodynamics and narrowing the field of the applicability of the theory. This is one of the reasons why such a statement of the problem permits one to consider a comparatively narrow class of thermodynamic systems.

The trend in the thermodynamic theory, being intensely developed in the last decade and known as "rational thermodynamics" [9-12], does not place before the theory such a far-reaching objective which would have restricted validity. Its task is to isolate a subclass of thermodynamically admissible equations from a vast class of constitutive equations that were described by some means or other. It turned out that the constructive role of thermodynamic principles, consisting in an appreciable restriction of the possible choice of constitutive equations, is extremely great – up to a reduction in the number of such equations. The task of removing such an arbitrariness that remains after the use of thermodynamic principles with a view of specifying constitutive equations remains outside the scope of thermodynamics and is solved either experimentally or within the confines of other theories (e.g., statistical). Owing to a rigorous formalization of the concepts used, precise formulation of the original premises, high degree of axiomatization, and to the use of the present-day mathematical methods, this trend has demonstrated its productivity in application to the most diversified thermodynamic systems, including those which could not be considered on the basis of former thermodynamic approaches, for example, to systems with memory. Methods were developed for investigating the properties of constitutive equations, resulting from thermodynamic limitations, for the most diversified thermodynamic systems, including the properties of relaxational functions of media with memory [13-20]. Owing to the variety of the conclusions of the general theory the possibility appeared for testing different versions of the initial postulates and for selecting the most adequate of them by comparing the conclusions of the theory with experiment. This stimulating factor was the reason for a deep analysis of the

---

Academic Scientific Complex "A. V. Luikov Heat and Mass Transfer Institute of the Academy of Sciences of Belarus," Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 68, No. 4, pp. 633-646, July-August, 1995. Original article submitted October 10, 1994.

1062-0125/95/6804-0518$12.50

foundations of the theory and investigations of its versions based on different formulations of the second law [21-27].

However, at the initial stage of the development this trend suffered from a drawback residing in the fact that in addition to the constitutive equations postulated in the theory for closing conservation laws, it was necessary to postulate still another constitutive equation for the thermodynamic potential (entropy, free energy, etc.), which was a prerequisite for the formulation of the second law. And while for the first type of equations there are justifications outside the thermodynamics (for example, from experiment or from the statistical theory), the constitutive equation for the nonequilibrium thermodynamic potential is unknown in the majority of cases, except for the simplest thermodynamic systems. And though, notwithstanding this, it was possible to obtain conclusions independent of the specific form of the thermodynamic potential, nevertheless, the absence of its determination should be considered a drawback of the theory.

In 1974 Coleman and Owen published their work [27] in which they developed some of the earlier ideas of Day [12] and worked out an extremely general thermodynamic theory using the principle of a fixed sign for a thermodynamic action in closed processes for the formulation of the second law. In this case the thermodynamic potential is not incorporated into the set of primary concepts and is a quantity, which is determined and constructed in the theory. Apart from overcoming the above deficiency, this approach also holds promise because it makes it possible to construct a constitutive equation for the thermodynamic potential that contains practically important information; the derivation of this equation can be considered as one of objectives of the thermodynamic theory. However, before getting down to the solution of this urgent problem, it is necessary to find out what conditions should be satisfied by a thermodynamic system for the fulfillment of the initial postulates that result in the existence of the thermodynamic potential. In other words, it is necessary to find the necessary and sufficient conditions for satisfying the second law in this formulation.

In the present study we solve this range of problems for generalized (in the sense in which the systems are introduced in [28]) linear thermodynamic systems with memory that model the most diversified physical systems but nevertheless relate to a certain specific realization of still more general Coleman-Owen systems.

1. Notation and Definitions. Let R and R^+ be sets of real and real nonnegative numbers; S is the configuration space representing a linear finite-dimensional vector space of the elements α, β, γ ... with scalar product $<\cdot>$, norm $|\cdot|$, and zero element 0; $L(S)$ is the vector space of all the linear mappings A, B, C, ... of the space S into itself with the norm

$$|[A]| = \sup\left\{ |\mathrm{A}\alpha| : \alpha \in S, \quad |\alpha| = 1 \right\}, \quad \mathrm{A} \in L(S) \tag{1.1}$$

and zero element **0**.

For any $\mathrm{A} \in L(S)$ the transformation in $L(S)$ *conjugate* to it (or the transposed A) is denoted by $\mathrm{A}^{\times}$ and is defined by

$$\langle \alpha, \mathrm{A}\beta \rangle = \langle \beta, \mathrm{A}^{\times}\alpha \rangle \quad \text{for all} \quad \alpha, \beta \in S. \tag{1.2}$$

The function of time ε: $R \to S$, called the *configurational trajectory* of the system, is a continuous function bounded on each finite interval, with a bounded derivative on finite intervals; for this function there is such t_0, that $\varepsilon(t) = \varepsilon_0$ for all $t \leq t_0$ (ε_0 is a fixed element from S).

The configurational history of the system up to the moment t is the function ε^t: $R^+ \to S$, defined as follows:

$$\varepsilon^t(s) = \varepsilon(t - s). \tag{1.3}$$

The differential configurational history up to the moment t is the function $\dot{\varepsilon}^t$: $R^+ \to S$:

$$\dot{\varepsilon}^t(s) = \frac{d}{dt}\varepsilon^t(s) = -\frac{d}{ds}\varepsilon(t - s). \tag{1.4}$$

The Hilbert space $\mathcal{H}$ of piecewise-continuous bounded functions f: $R^+ \to S$ with a compact carrier and a finite norm

$$\| f \| = \left(\int_0^\infty |f(s)|^2 \xi(s)\, ds \right)^{1/2} \tag{1.5}$$

will be called a *space of differential histories.* In Eq. (1.5) $\xi > 0$ is the influence function. It is assumed to be continuous, integrable on R^+, vanishing nowhere, and having the property

$$\lim_{s \to \infty} \frac{\xi(s)}{\xi(s+T)} < \infty . \tag{1.6}$$

The *state* Λ is the pair $\Lambda = \{\alpha, f\}$, where $\alpha \in S$, $f \in \mathcal{H}$, and the set of all of such pairs with the norm $\| \cdot \|$

$$\| \Lambda \|_s = (|\alpha|^2 + \| f \|^2)^{1/2} \tag{1.7}$$

forms *the space of states* $\mathcal{H}$.

For a specified configurational trajectory $\varepsilon(\cdot)$ and arbitrary moment t *the state of the system at the moment* t is defined as

$$\Lambda^t = \left\{ \varepsilon(t) , \dot{\varepsilon}^t \right\} . \tag{1.8}$$

The equilibrium state is $\Lambda^+ = \{\alpha, 0^+\}$, where $\alpha \in S$, $0^+ \in \mathcal{H}$ and $0^+(s) = 0$ for all $s \in R^+$.

The relaxation function $\mathbf{R}$: $R^+ \to L(S)$ is a continuous twice-differentiable function such that

$$\int_0^\infty |[\mathrm{R}(s)]|\, s\, ds < \infty \tag{1.9}$$

and, consequently:

$$\mathrm{R}(\infty) = 0 \quad \text{and} \quad \int_0^\infty |[\mathrm{R}(s)]|\, ds < \infty . \tag{1.10}$$

Associated with the relaxation function is the linear function of state, i.e., *the constitutive functional of generalized forces* $\hat{\sigma}$: $\mathbb{S} \to S$, which for the state $\Lambda = \{\alpha, f\}$ is defined as

$$\hat{\sigma}(\Lambda) = \hat{\sigma}(\alpha , f) = \sigma_0 + \mathrm{E}\alpha + \int_0^\infty \mathrm{R}(s) f(s)\, ds . \tag{1.11}$$

For each configurational trajectory of the system $\varepsilon(t)$, with the help of the functional (1.11) it is possible to unambiguously determine the trajectory of the generalized forces σ_ε: $R \to S$:

$$\sigma_\varepsilon(t) = \hat{\sigma}(\Lambda^t) = \hat{\sigma}(\varepsilon(t), \dot{\varepsilon}^t) = \sigma_0 + \mathrm{E}\varepsilon(t) + \int_0^\infty \mathrm{R}(s)\, \dot{\varepsilon}(t-s)\, ds . \tag{1.12}$$

The thermodynamic trajectory is the pair $\{\varepsilon(t), \sigma_\varepsilon(t)\}$ consisting of the configurational trajectory and the trajectory corresponding to it, i.e., σ_ε: $R \to S$ of the generalized forces.

The process of duration $T (T > 0)$ is the function h: $(0, T] \to S$, bounded and piecewise-continuous, with which the transformation P_h^T: $\mathbb{S} \to \mathbb{S}$ is associated in the space of states defined as follows: for any $\Lambda = \{\alpha, f\} \in \mathbb{S}$

$$P_h^T \Lambda = \Lambda_{(h)} = \left\{ \alpha_{(h)} , f_{(h)} \right\} , \tag{1.13}$$

where

$$\alpha_{(h)} = \alpha + h^i(T) , \tag{1.14}$$

$$f_{(h)}(s) = \begin{cases} f(s-T) & \text{for} \quad s \in [T , \infty) ; \\ h(T-s) & \text{for} \quad s \in [0 , T) ; \end{cases} \tag{1.15}$$

$$h^i(t) = \int_0^t h(s)\, ds\,, \tag{1.16}$$

P_h^T is the transformation induced by the process h. We state that the process h transforms the system from the initial state Λ into the final state $P_h^T\Lambda$.

The set of all the processes and the set of the processes of duration T will be denoted by $\mathscr{P}$ and $\mathscr{P}_T$, respectively. The process $u(s) = 0$ for all $S \in (0, T)$ will be called a *fixed process* of duration T.

It can be easily shown that the transformation P_h^T associated with the process h takes the system from the state at the moment t $\Lambda^t = \{\varepsilon(t), \varepsilon^t\}$, corresponding to the deformational trajectory $\varepsilon(\tau)$, to the state at the moment $t+T$, corresponding to the deformational trajectory $\varepsilon_h(\tau)$, which is defined on $(-\infty, t+T)$ as follows:

$$\varepsilon_h(\tau) = \begin{cases} \varepsilon(\tau) & \text{for} \quad \tau \le t\,, \\ \varepsilon(t) + h^i(\tau - t) & \text{for} \quad \tau \in (t\,,\ T + t\,]\,. \end{cases} \tag{1.17}$$

It can be shown that due to the above assumptions, for any process h the transformation P_h^T associated with it is continuous in $\mathbb{S}$.

The composition of the processes h_1 of duration T_1 and h_2 of duration T_2 is the process $h_1 \circ h_2$ of duration T_1+T_2 defined in the following way:

$$h_1 \circ h_2 = \begin{cases} h_1(\tau) & \text{for} \quad \tau \in (0\,,\ T_1\,]\,, \\ h_2(\tau - T_1) & \text{for} \quad \tau \in (T_1\,,\ T_1 + T_2)\,. \end{cases} \tag{1.18}$$

We can determine the composition $h_2 \circ h_1$ in a similar way. It can be easily shown that

$$P_{h_1 \circ h_2}^{T_1 + T_2}\Lambda = P_{h_2}^{T_2} P_{h_1}^{T_1}\Lambda\,.$$

We can also determine for any $\tau \le T$ the reduction of the process of duration T on the interval $(0, \tau)$ by contracting the region of the determination of the process h on $(0, \tau)$; this will be the process of duration τ. The transformation P_h^T associated with it will be determined appropriately.

In concluding this section, we introduce the following notation. Let S^* be the linear vector space above the field of complex numbers such that its contraction on the field of real numbers coincides with S; then $L^*(S^*)$ is the vector space of linear mappings from S^* into S^*. For any element α from S^* the complex conjugate is denoted by α^*. If $\mathbf{R}$: $R^+ \to L(S)$ is the relaxation function, then r: $R^+ \to R^+$, $\tilde{r}$: $R^+ \to R^+$, $\bar{r}$: $R^+ \to R^+$ and the *expanded relaxation function* $\mathring{\mathbf{R}}$: $R \to L(S)$ are defined by the relations

$$r(s) = |\,[\mathbf{R}(s)]\,|\,, \tag{1.19}$$

$$\tilde{r}(s) = \sup\left\{ r(\lambda)\ |\lambda \in [s\,,\ \infty) \right\}, \tag{1.20}$$

$$\bar{r}(s) = \int_s^\infty r(\lambda)\, d\lambda\,, \tag{1.21}$$

$$\mathring{\mathbf{R}}\ (s) = \begin{cases} \mathbf{R}(s) & \text{for} \quad s \in (0\,,\ \infty)\,, \\ (\mathbf{R}(0) + \mathbf{R}^\times(0))/2 & \text{for} \quad s = 0\,, \\ \mathbf{R}^\times(-s) & \text{for} \quad -s \in (0\,,\ \infty)\,. \end{cases} \tag{1.22}$$

The operations on the functions in (1.21) and (1.22) can be used once more; for example, $\bar{\tilde{r}}$ is understood to represent

$$\bar{\tilde{r}}(s) = \int_s^\infty \tilde{r}(\lambda)\, d\lambda\,. \tag{1.23}$$

It is obvious that the functions generated by operations (1.21) and (1.22) are monotonous by definition.

We will use the symbol $\mathbf{R}_L$ for the Laplace transformation of $\mathbf{R}$, $\mathbf{R}_c$ for the Fourier cosine-transformation of $\mathbf{R}$, $\mathbf{R}_s$ for the Fourier sine-transformation of $\mathbf{R}$, and $\overset{\circ}{\mathbf{R}}_F$ for the Fourier transformation of $\overset{\circ}{\mathbf{R}}$:

$$\mathbf{R}_L(p) = \int_0^\infty \mathbf{R}(s)\, e^{-sp}\, ds\,, \tag{1.24}$$

$$\mathbf{R}_c(\omega) = \int_0^\infty \mathbf{R}(s) \cos(\omega s)\, ds\,, \tag{1.25}$$

$$\mathbf{R}_s(\omega) = \int_0^\infty \mathbf{R}(s) \sin(\omega s)\, ds\,, \tag{1.26}$$

$$\overset{\circ}{\mathbf{R}}_F(\omega) = \int_{-\infty}^{\infty} \overset{\circ}{\mathbf{R}}(s)\, e^{-i\omega s}\, ds\,, \tag{1.27}$$

where $\mathbf{R}_L$: $C \to L^*(S^*)$, $\mathbf{R}_c$: $R^+ \to L(S)$, $\mathbf{R}_s$: $R^+ \to L(S)$, $\overset{\circ}{\mathbf{R}}_F$: $R \to L^*(S^*)$.

Thermodynamic Postulates and Corollaries.. *The action* (or thermodynamic action) performed by the process h of duration T from the state Λ is the function a: $\mathbb{S} \times \mathcal{P} \to R$, which is defined as follows:

$$a(\Lambda,\, h) = \int_0^T \langle \hat{\sigma}(P_h^\tau \Lambda)\,,\, h(\tau) \rangle > d\tau\,, \tag{2.1}$$

where P_h^T is the transformation associated with the reduction of the process h onto the interval $(0, \tau)$. It is evident that the action is continuous in $\mathbb{S}$ at a fixed h and, moreover, it is additive on the composition of the two processes h_1 and h_2 of duration T_1 and T_2 in the sense that

$$a(\Lambda,\, h_1 \circ h_2) = a(\Lambda,\, h_1) + a(P_{h_1}^{T_1} \Lambda,\, h_2) =$$

$$= \int_0^{T_1} \langle \hat{\sigma}(P_{h_2}^\tau \Lambda)\,,\, h_1(\tau) \rangle\, d\tau + \int_0^{T_2} \langle \hat{\sigma}(P_{h_2}^\tau P_{h_1}^{T_1} \Lambda)\,,\, h_1(\tau) \rangle\, d\tau\,. \tag{2.2}$$

In specific physical applications of this theory, there is an integral of reduced heat for the thermodynamic action. All the concepts and definitions formulated above is a certain, more specific realization of the abstract mathematical theory of Coleman-Owen's thermodynamic systems [27], so that there is quite a good correspondence between the basic concepts and postulates of the both theories.

We shall go over to the formulation of the postulates of the thermodynamic theory.

The postulate that expresses the second law of thermodynamics in the Coleman-Owen sense [27] will be formulated as follows.

P1. *At any initial state* $\Lambda \in \mathbb{S}$ *the action a has the following property: for any* $\epsilon > 0$ *there is such* $\delta > 0$ *such that if* $h \in \mathcal{P}^{\mathrm{T}}$ *and*

$$\| \Lambda - P_h^T \Lambda \|_s < \delta\,, \tag{2.3}$$

then

$$a(\Lambda,\, h) > -\epsilon\,. \tag{2.4}$$

Speaking nonstrictly, this postulate means that if a certain process takes a system to a rather small neighborhood of the initial state, then the action performed in this process will be nonnegative with any high accuracy. This statement is a generalization and a strict mathematical formalization of the formulation of the second law used in classical thermodynamics requiring nonnegativeness of the reduced heat integral in a cyclic process [1].

We shall consider that the relaxation function **R** is *coordinated* with the influence function ξ if there is $M > 0$ such that

$$\bar{r}(s) \leq M\xi(s) \quad \text{for all} \quad s \in R^{+}. \tag{2.5}$$

In the theory suggested, between the relaxation function and the influence function a certain interrelation is assumed which is expressed in the following postulate, which, without any additional reservations, will be considered satisfied throughout in what follows.

P2. *The influence function ξ is coordinated with the relaxation function* **R**.

At first glance this postulate has no relation to thermodynamics and can be considered as a technical assumption in the mathematical apparatus used. In actual fact it establishes a certain interrelation between the postulated constitutive equation (1.11), whose representative is **R**, and the thermodynamic potential, which is constructed in the theory and which possesses, as will be seen from what follows, certain properties of continuity in the space $\mathbb{S}$, whose representative is ξ. For this reason, this postulate is included into a series of thermodynamic potentials.

The basic corollary of the second law of thermodynamics is the existence of a thermodynamic potential and the fulfillment of the Clausius-Duhem inequality, which forms the content of the next theorem (this is an analog of Theorem 3.3 of [27] for the class of thermodynamic systems considered).

Theorem 1. *There is a function of state $\hat{\psi}: \mathbb{S} \to R$ (thermodynamic potential), definite and continuous on $\mathbb{S}$, so that for any $\Lambda \in \mathbb{S}$ and any $h \in \mathscr{P}$ the following inequality is satisfied:*

$$\hat{\psi}(P_h\Lambda) + \hat{\psi}(\Lambda) \leq a(\Lambda, h). \tag{2.6}$$

The theorem is proved on the basis of arguments similar to those used in [27] with only one small modification: in [27] a weaker version of **P1** is used and only the upper semicontinuity of the potential $\hat{\psi}$ is proved. The upper semicontinuity in the state Λ means the following: for any $\epsilon > 0$ there exists $\delta > 0$ such that from

$$\Lambda' \in \mathbb{S} \quad \text{and} \quad \| \Lambda - \Lambda' \|_s < \delta \tag{2.7}$$

it follows that

$$\hat{\psi}(\Lambda') - \hat{\psi}(\Lambda) < \epsilon. \tag{2.8}$$

Since here in **P1** the property of action a indicated in it was assumed to be satisfied in any state, then $\hat{\psi}$ is upper semicontinuous and also in state Λ; therefore there is $\delta > 0$ such that (2.7) yields not only (2.8), but also

$$-\hat{\psi}(\Lambda') + \hat{\psi}(\Lambda) < \epsilon.$$

From this the continuity of $\hat{\psi}$ in any state follows.

This result is fundamentally important, since it demonstrates the interrelation between the formulation of the second law used here and other well-known formulations. Its applications are also not unimportant; for example, it can be used to solve in principle the urgent problem of finding the dissipation function (functional) for media with memory from the well-known constitutive equations for the internal energy and stress tensor, as well as deriving thermodynamic restrictions of constitutive equations.

The proof of this theorem is constructive, i.e., it contains the definition of the functional $\hat{\psi}$. According to [27], it is determined accurate to its value in a certain surveyed (for example, equilibrium) state Λ_0^{+} and is specified as follows:

$$\hat{\psi}(\Lambda) = \sup\left\{ \inf\left\{ a(\Lambda_0^{+}, h) \mid h \in \mathscr{P}, \ \| P_h\Lambda_0^{+} - \Lambda \|_s < \epsilon \right\} \mid \epsilon > 0 \right\}. \tag{2.9}$$

However, in [29] it is shown that for many thermodynamic systems there exists an infinite set of potentials among which we may distinguish the maximum and minimum ones. In view of this, there arises the problem of obtaining explicit expressions for these potentials in the case of specific thermodynamic system and elucidating their physical meaning. We shall refer to this problem in the last article of the series. But first we shall attempt to find the necessary and sufficient conditions that must be satisfied by the constitutive equation (1.11) in order for the second law to be fulfilled in the form of **P2**. This means that for the given equation we shall find a full set of thermodynamic restrictions that follows from the second law.

To prove the basic theorem containing these conditions (this will be given in the next article), we will need certain auxiliary purely mathematical statements that form the content of the forthcoming lemmas.

Lemma 1. *The following conditions are equivalent:*

i) for any locally quadratically integrable function g: $R^+ \to S$ and any $T > 0$ the relaxation function **R** *satisfies the inequality*

$$\int_0^T \int_0^t \langle g(t)\, ,\ \mathbf{R}(t-s)\, g(s) \rangle\, ds\, dt \geq 0\, ; \tag{2.10a}$$

ii) for any $\omega \in R$ and any $\alpha, \beta \in S^*$ the expanded relaxation function $\mathring{\mathbf{R}}$ (determined in (1.22) satisfies the inequality

$$\langle \alpha^*\, ,\ \mathring{\mathbf{R}}_F(\omega)\, \alpha \rangle \geq 0\, ; \tag{2.10б}$$

iii) for any $\mu \in R^+$, $\omega \in R$, $\alpha, \beta \in S^*$ the relaxation function **R** *satisfies the inequality*

$$\langle \alpha^*\, ,\ \Big(\mathbf{R}_L(\mu + i\omega) + \mathbf{R}_L^{\times}(\mu - i\omega) \Big)\, \alpha \rangle \geq 0\, . \tag{2.10в}$$

Lemma 2. *If the relaxation function* **R** *satisfies the conditions of lemma 1, then it satisfies the inequality*

$$\int_0^T \int_0^t \langle g(t)\, ,\ \mathbf{R}(t-s)\, g(s) \rangle\, ds\, dt + \sum_{k=1}^{\infty} \int_0^T \int_0^t \langle g(t)\, ,\ \mathbf{R}(t - s + kT)\, g(s) \rangle\, ds\, dt \geq 0 \tag{2.11}$$

for any locally quadratically integrable g: $R^+ \to S$ and any $T > 0$.

The following two lemmas result from **P1**.

Lemma 3. *There exists $M_1 > 0$ such that for all $s \geq 0$*

$$\tilde{r}(s) \leq M_1 \xi(s)\, . \tag{2.12}$$

Lemma 4. *For any $T > 0$ there exists $M_{(t)} > 0$ such that for all $s \geq T$*

$$\bar{r}(s - T) \leq M_{(T)} \xi(s)\, . \tag{2.13}$$

The proofs of all the lemmas are given in the Appendix.

APPENDIX

Proof of Lemma 1. The equivalence between *i)* and *iii)* is an actually well-known result in the theory of linear dynamic systems [30]. The condition *iii)* coincides with the statement that $\mathbf{R}_L$ is a positive real function (in the terminology of [30]), and the condition i) is equivalent to the requirement of the passive state for a dynamic input-output type system representable by the pulse function of response **R** with input $\dot{\varepsilon}$ and output σ (coupled, as in (1.12), on the condition that $\sigma_0 = 0$, $\mathbf{E} = 0$). According to Theorem 1 from [30], these conditions are equivalent.

To complete the proof of the lemma, it is necessary to prove the equivalence between *ii)* and *iii)*. Assuming $\mu = 0$ in (2.10c) and using definition (1.23), we can easily show that *ii)* follows from *iii)*. Conversely, let us assume that *ii)* is satisfied and, consequently, inequality (2.10b) is valid. The left-hand side of inequality (2.10c) can be represented in the form

$$\langle \alpha^*\, ,\ \left(\int_0^{\infty} \mathbf{R}(s)\, e^{-\mu s}\, e^{-i\omega s}\, ds + \int_0^{\infty} \mathbf{R}^{\times}(s)\, e^{-\mu s}\, e^{i\omega s}\, ds \right) \alpha \rangle = \int_{-\infty}^{\infty} \langle \alpha^*\, ,\ \mathring{\mathbf{R}}(s)\, \alpha \rangle\, \varphi(s)\, e^{-i\omega s}\, ds\, , \tag{A.1}$$

where

$$\varphi(s) = \mathrm{e}^{-\mu\,|s|}\,. \tag{A.2}$$

The Fourier transform of this function is

$$\varphi_F(\omega) = \frac{\mu}{\mu^2 + \omega^2}\,. \tag{A.3}$$

Since in *iii)* it was assumed that $\mu \geq 0$, from (A.3) we have

$$\varphi_F(\omega) \geq 0 \tag{A.4}$$

for all $\omega \in R$.

Using the Borel theorem concerning the convolution, the right-hand side of (2.10c) in (A.1) can be transformed to

$$\langle \alpha^*\,,\ \left(\mathbf{R}_L(\mu + i\omega) + \mathbf{R}_L^{\times}(\mu - i\omega)\right)\alpha\,\rangle = \int_{-\infty}^{\infty} \langle\, \alpha^*\,,\ \overset{\circ}{\mathbf{R}}_F(\omega - \omega_0)\,\alpha\,\rangle\, \varphi_F(\omega_0)\, d\omega_0\,. \tag{A.5}$$

Now, *iii)* follows directly from (A.5) by virtue of *ii)* and (A.4), as was to be shown.

2. Proof of Lemma 2. With the aid of the function g from (2.11) we define the function g_T with the carrier limited on $[0, T]$ and the function $g_{(T)}$, periodic on R, with the period T in the following manner:

$$g_T(s) = \begin{cases} g(s) & \text{for} \quad s \in [0\,,\ T]\,, \\ 0 & \text{for the remaining} \quad s\,; \end{cases} \tag{A.6}$$

$$g_{(T)}(s + kT) = g(s) \quad \text{for} \quad s \in [0\,,\ s)\,, \quad k = 0\,,\ \pm 1\,,\ \pm 2\,,\ldots \tag{A.7}$$

With the help of $g_{(T)}$ and $\overset{\circ}{\mathbf{R}}$ introduced into (1.22), the left-hand side of inequality (2.11), which will be denoted by I, can be transformed as follows:

$$I = \int_0^T \int_0^t \langle\, g(t)\,,\ \mathbf{R}(t - s)\, g(s)\,\rangle\, ds\, dt + \sum_{k=1}^{\infty} \int_0^T \int_0^T \langle\, g(t)\,,\ \mathbf{R}(t - s + kT)\, g(s)\,\rangle\, ds\, dt =$$

$$= \frac{1}{2} \left(\int_0^T \int_0^t \langle\, g(t)\,,\ \mathbf{R}(t - s)\, g(s)\,\rangle\, ds\, dt + \int_0^T \int_t^T \langle\, g(t)\,,\ \mathbf{R}^{\times}(s - t)\, g(s)\,\rangle\, ds\, dt + \right.$$

$$\left. + \int_T^{\infty} \int_0^T \langle\, g_{(T)}(t)\,,\ \mathbf{R}(t - s)\, g(s)\,\rangle\, ds\, dt + \int_0^T \int_{-\infty}^0 \langle\, g(t)\,,\ \mathbf{R}(t - s)\, g_{(T)}(s)\,\rangle\, ds\, dt \right) =$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \langle\, g_{(T)}(t)\,,\ \int_0^T \overset{\circ}{\mathbf{R}}(t - s)\, g(s)\, ds\,\rangle\, dt\,. \tag{A.8}$$

Let us represent the function $\overset{\circ}{\mathbf{R}}$ in terms of its Fourier transform (1.27):

$$\overset{\circ}{\mathbf{R}}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overset{\circ}{\mathbf{R}}_F(\omega)\, \mathrm{e}^{i\omega s}\, d\omega\,, \tag{A.9}$$

and the periodic function $g_{(T)}$ in terms of its Fourier series:

$$g_{(T)}(t) = \sum_{k=-\infty}^{\infty} c_k \exp\left(\frac{i\,2\pi k}{T}\, t\right)\,,$$

$$c_k = c_{-k}^* = \frac{1}{2} \int_0^T g(s) \exp\left(\frac{i\,2\pi k}{T}\, s\right) ds\,. \tag{A.10}$$

Substituting (A.9), (A.10) into (A.8), we obtain

$$I = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle \sum_{k=-\infty}^{\infty} c_k \exp\left(\frac{i\,2\pi k}{T} t \right) ,\ \int_0^T \left(\int_{-\infty}^{\infty} \overset{\circ}{\mathbf{R}}_F(\omega)\, e^{i\omega(t-s)}\, d\omega \right) g(s)\, ds \right\rangle dt =$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle \sum_{k=-\infty}^{\infty} c_k \exp\left(\frac{i\,2\pi k}{T} t \right) ,\ \int_{-\infty}^{\infty} \overset{\circ}{\mathbf{R}}_F(\omega)\, e^{i\omega t} \int_0^T g(s)\, e^{-i\omega s}\, ds\, d\omega \right\rangle dt =$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle \sum_{k=-\infty}^{\infty} c_k \exp\left(\frac{i\,2\pi k}{T} t \right) ,\ \int_{-\infty}^{\infty} \overset{\circ}{\mathbf{R}}_F(\omega)\, g_{TF}\,\omega\, e^{i\omega t}\, d\omega \right\rangle dt, \tag{A.11}$$

where $g_{(TF)}$ is the Fourier-transform of the function g_T introduced in (A.6).

The change of the order of integration used in (A.11) is valid, since the function $\overset{\circ}{\mathbf{R}}_F(\omega) g(s) e^{-i\omega s}$ is absolutely integrable on $[0, T] \times R$ by virtue of the absolute integrability of $\overset{\circ}{\mathbf{R}}_F$, which is obvious, since $\mathbf{R}$ was assumed to be continuously twice-differentiable. The series in the last term in (A.11) can be integrated term by term, since it represents the expansion of a bounded function. Performing this integration, we obtain

$$I = \frac{1}{4\pi} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle c_k ,\ \int_{-\infty}^{\infty} \overset{\circ}{\mathbf{R}}_F(\omega)\, g_{TF}(\omega) \exp\left(i \left(\frac{2\pi k}{T} + \omega \right) t \right) d\omega\, dt \right\rangle =$$

$$= \frac{1}{4\pi} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle g_{TF}(\omega) ,\ \overset{\circ}{\mathbf{R}}{}_F^{\times}(\omega)\, c_k \right\rangle \exp\left(i \left(\frac{2\pi k}{T} + \omega \right) t \right) d\omega\, dt. \tag{A.11a}$$

With the use of the obvious relations

$$\overset{\circ}{\mathbf{R}}{}_F^{\times}(\omega) = \overset{\circ}{\mathbf{R}}_F(-\omega) \quad \text{and} \quad g_{TF}(-\omega) = \overset{*}{g}_{TF}(\omega)$$

Eq. (A.11a) can be reduced to

$$I = \frac{1}{4\pi} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle \overset{*}{g}_{TF}(\omega) ,\ \overset{\circ}{\mathbf{R}}_F(\omega)\, c_k \right\rangle \exp\left(i \left(\frac{2\pi k}{T} - \omega \right) t \right) d\omega\, dt. \tag{A.12}$$

Recall now the definition and properties of the Fourier integral [31]: if the function φ is absolutely integrable on R and satisfies the Dini condition, then the following relation is satisfied:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\nu) \exp\left(-i(\nu - x)\lambda\right) d\nu\, d\lambda = \varphi(x). \tag{A.13}$$

The function $\langle \overset{*}{g}_{TF}(\omega), \overset{\circ}{\mathbf{R}}_F(\omega) c_k \rangle$ in (A.12) is absolutely integrable on R by virtue of the absolute integrability of $\overset{\circ}{\mathbf{R}}_F$ and boundedness of g_{TF}. Moreover, $\overset{*}{g}_{TF}$ and $\overset{\circ}{\mathbf{R}}_F$ are continuous and differentiable. This means that it satisfies the Dini condition, and we must use (A.13) in (A.12). As a result we have

$$I = \frac{1}{2} \sum_{k=-\infty}^{\infty} \left\langle \overset{*}{g}_{TF}\left(\frac{2\pi k}{T} \right) ,\ \overset{\circ}{\mathbf{R}}_F\left(\frac{2\pi k}{T} \right) c_k \right\rangle. \tag{A.14}$$

We note that according to (A.10)

$$g_{TF}\left(\frac{2\pi k}{T} \right) = T c_k , \tag{A. 15}$$

so that we can rewrite (A.14) as

$$I = \frac{T}{2} \sum_{k=-\infty}^{\infty} \left\langle \overset{*}{c}_k ,\ \overset{\circ}{\mathbf{R}}_F\left(\frac{2\pi k}{T} \right) c_k \right\rangle. \tag{A.16}$$

According to the condition *ii*) of Lemma 1, all the terms of the series in (A.16) are nonnegative and, consequently, Lemma 2 has been proved.

Proof of Lemma 3. As is stated in **P2**, there is $M > 0$ such that

$$\bar{\bar{r}}(s) \leq M\xi(s) \quad \text{for all} \quad s \in R^+ . \tag{A.17}$$

Let us use the definition of the function $\bar{\bar{r}}$ (1.23) and the fact that, according to the definition of (1.20), the function $\tilde{r}$ is always positive and monotonously decreasing:

$$\bar{\bar{r}}(s) = \int_s^\infty \tilde{r}(\lambda)\, d\lambda = \int_s^{s+\tau_0} \tilde{r}(\lambda)\, d\lambda + \int_{s+\tau_0}^\infty \tilde{r}(\lambda)\, d\lambda \geq \int_s^{s+\tau_0} \tilde{r}(\lambda)\, d\lambda \geq \tau_0 \tilde{r}(s+\tau_0) . \tag{A.18}$$

Here $\tau_0 > 0$ is an arbitrarily selected fixed number. Let us introduce the designation

$$M_{\tau_0} \overset{\text{def}}{=} \sup \left\{ \frac{\xi(s)}{\xi(s+\tau_0)} \,\middle|\, s \in (0, \infty) \right\} . \tag{A.19}$$

According to the properties of the influence function, specified in Section 1, this supremum exists and is finite. It is evident that

$$\xi(s) \leq M_{\tau_0} \xi(s+\tau_0) \quad \text{for all} \quad s \in R^+ . \tag{A.20}$$

Combining (A.17), (A.18), and (A.20), we obtain

$$\tilde{r}(s+\tau_0) \leq \frac{M M_{\tau_0}}{\tau_0} \xi(s+\tau_0) \quad \text{for all} \quad s \in R^+ ,$$

or in alternative form

$$\tilde{r}(s) < \frac{M M_{\tau_0}}{\tau_0} \xi(s) \quad \text{for all} \quad s \in [\tau_0, \infty) . \tag{A.21}$$

Using the notation

$$m_{\tau_0} \overset{\text{def}}{=} \sup \left\{ \frac{\tilde{r}(s)}{\xi(s)} \,\middle|\, s \in [0, \tau_0) \right\} , \tag{A.22}$$

we can write

$$\tilde{r}(s) \leq m_{\tau_0} \xi(s) \quad \text{for all} \quad s \in [0, \tau_0) . \tag{A.23}$$

Using the notation

$$M_1 \overset{\text{def}}{=} \max \left(m_{\tau_0} , \frac{M M_{\tau_0}}{\tau_0} \right) \tag{A.24}$$

we can combine (A.21) and (A.23):

$$\tilde{r}(s) \leq M_1 \xi(s) \quad \text{for all} \quad s \in [0, \infty) ,$$

as was to be shown.

Proof of Lemma 4. Since by virtue of definitions (1.19), (1.20)

$$r(s) \leq \tilde{r}(s) \quad \text{for all} \quad s \in R^+ , \tag{A.25}$$

the same inequality is also valid for the integrals of these functions (see designations (1.21), (1.23)):

$$\bar{r}(s) \leq \bar{\bar{r}}(s) \quad \text{for all} \quad s \in R^+ . \tag{A.26}$$

For any $T > 0$ from (A.20) we have

$$\xi(s) \leq M_T \xi(s+T) \quad \text{for all} \quad s \in R^+ . \tag{A. 27}$$

Substituting (1.26) and (1.27) into (1.17) and introducing a new variable $\tau = s+T$, we have

$$\bar{r}(\tau - T) \leq M M_T \xi(\tau) ,$$

and this is equivalent to (2.13) with $M_{(T)} = MM_T$, as was to be shown.

This work was financed by the Fund for Fundamental Investigations of the Republic of Belarus (project T20-359).

REFERENCES

1. S. R. De Groot and P. Mazur, Nonequilibrium Thermodynamics [Russian translation], Moscow (1966).
2. P. Glensdorf and I. Prigogine, Thermodynamic Theory of Structure, Stability, and Fluctuations [Russian translation], Moscow (1979).
3. R. L. Stratonovich, Nonlinear Nonequilibrium Thermodynamics [in Russian], Moscow (1985).
4. I. Diarmati, Nonequilibrium Thermodynamics. Theory of Field and Variational Principles [Russian translation], Moscow (1974).
5. L. A. Kamenyarzh and L. I. Sedov, PPM, **43**, Issue 1, 1-3 (1979).
6. A. A. Il'yushin and G. A. Il'yushina, Vestn. Mosk. Univ., Ser. Mat. Mekh., No. 3, 73-80 (1983).
7. I. P. Vyrodov, Zh. Fiz. Khim., **59**, Issue 10, 2385-2399 (1985).
8. I. F. Bakhareva, Nonlinear and Nonequilibrium Thermodynamics [in Russian], Saratov (1976).
9. C. Truesdell and W. Noll, in: Handbuch der Physik, III/3 (1965), pp. 1-602.
10. C. Truesdell, Rational Thermodynamics, Berlin (1976).
11. C. Tuesdell, Mekh., No. 4, 99-136 (1970).
12. U. A. Day, Thermodynamics of Simple Media With Memory [Russian translation], Moscow (1974).
13. B. D. Coleman, Arch. Rat. Mech., Anal., **17**, No. 1, 1-45 (1964).
14. M. E. Gurtin, Arch. Rat. Mech. Anal., **28**, No. 1, 40-50 (1968).
15. B. D. Coleman and D. R. Owen, Arch. Rat. Mech. Anal., **36**, No. 4, 245-269 (1970).
16. P. J. Chen and J. W. Nunziato, ZAMP, **25**, 791-795 (1974).
17. V. L. Kolpashchikov and A. I. Shnipp, Int. J. Eng. Sci., **16**, No. 5, 503-514 (1978).
18. V. L. Kolpashchikov and A. I. Shnipp, Int. J. Eng. Sci., **21**, No. 5, 543-553 (1983).
19. V. L. Kolpashchikov and A. I. Shnip, Theoretical and Applied Mechanics, 4th Nat. Congress (Varna, 1981), Papers, Vol. 1, Sofia (1981).
20. V. L. Kolpashchikov and A. I. Shnip, Thermodynamics and Models of Nonclassical Media (Preprint No. 9 of the Heat and Mass Transfer Institute of the Academy of Sciences of BSSR), Minsk (1986).
21. M. E. Gurtin and W. O. Williams, Arch. Rat. Mech. Anal., **26**, No. 2, 83-117 (1967).
22. A. E. Green and P. M. Naghdi, Arch. Rat. Mech. Anal., **48**, No. 5, 352-378 (1972).
23. A. E. Green and A. Laws, Arch. Rat. Mech. Anal., **45**, No. 1, 47-53 (1972).
24. I. Muller, Arch. Rat. Mech. Anal., **26**, No. 2, 118-141 (1967).
25. I. Muller, Pure Appl. Chem., **20**, Nos. 3-4, 335-342 (1970).
26. J. Meixner, Arch. Rat. Mech. Anal., **33**, No. 1, 33-53 (1969).
27. B. D. Coleman and D. R. Owen, Arch. Rat. Mech. Anal., **54**, No. 1, 1-102 (1974).
28. V. L. Kolpashchikov and A. I. Shnip, Inzh.-Fiz. Zh., **65**, No. 2, 192-197 (1993).
29. B. D. Coleman and D. R. Owen, Arch. Rat. Mech. Anal., **58**, No. 1, 26-51 (1975).
30. J. Willems, J. of Franklin Inst., **301**, No. 6, 605-621 (1978).
31. A. N. Kolmogorov and S. V. Fomin, Elements of the Theory of Functions and of Functional Analysis [in Russian], Moscow (1976).